Simulation in Produktion
und Logistik 2021
Jörg Franke & Peter Schuderer (Hrsg.)
Cuvillier Verlag, Göttingen 2021

Analytical and Empirical Study of Proper Parameters for Theory of Constraints under Uncertainty

Analytische und empirische Studie geeigneter Parameter für die Engpasstheorie bei Unschärfe

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Abstract: This paper discusses stochastic analysis and defining parameters of drum-buffer-rope (DBR) by way of a reorder point system, in particular, the relationship between reorder point and cycle service level (CSL), as well as fill rate (FR). Analytical solutions are verified and validated experimentally through simulation. Literature typically discusses batch-oriented consumption, in reality however piece wise consumption with processing time per piece is more typically for production. There is no simple conversion between the two, but either requires a different experiment setup altogether. The difference between CSL, and FR is discussed next. The relationship between FR and reorder point proves complex, in particular in light of distribution functions other than normal distribution. In addition to variation in consumption, the model is then extended to supply variation.

1 Introduction

The Theory of Constraints (TOC) (Goldratt and Cox, 2004) is extensively discussed and analyzed in literature (Cox, J. and Schleier, J. 2010; Lödding, 2016) This paper extends the discussion and tries to understand TOC better from a stochastic perspective.

Originally, TOC has been conceived as a means for robust production control. The core problem of robustness in production control results from variation in almost all variables involved: demand, execution time, yield in execution and supply, capacity and availability, and so forth. Therefore, parameter selection in TOC must take the major sources of variability into account.

The aim of this paper is to study and understand these stochastic effects better and to provide mathematical solutions for optimal setup, in particular with regard to the selection of a proper reorder-point.

It turns out that not all effects are easy to describe and some even require empirical analysis based on simulation results, where mathematic rigor for the lack of proper models at hand is substituted by empirical outcomes.

2 Problem Description

Dimensioning of inventory management systems always poses a dilemma between availability and cost. Any parameter selection will always require a decision within this dilemma (Lödding, 2016). Literature discusses this extensively, usually though limited to steady state condition. TOC as an example of a reorder-point system with respect to stochastic variation deserves detailed discussion.

2.1 TOC and DBR

The general approach for TOC is shown in Figure 1: A chain of process steps is lined up. The process step with the lowest output rate is the bottleneck. By definition, there will always be such bottleneck. Throughput from the upstream process steps relative to the bottleneck is loading up any buffer in front of it, and the bottleneck throughput limitation is starving the downstream process steps. So the output from the complete process chain is limited by the throughput of the bottleneck. Then it is prudent to synchronize the upstream processes with the bottleneck, otherwise the before mentioned buffer will quickly overflow.

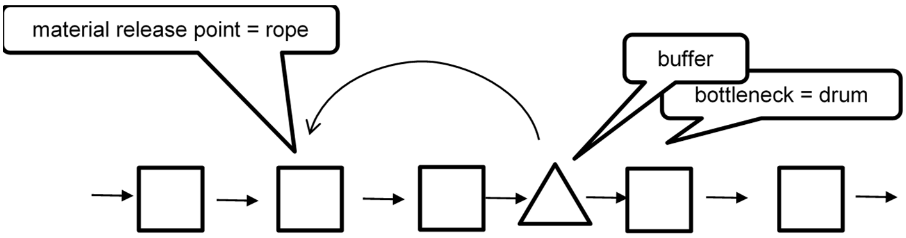

Figure 1: *General TOC Approach with DBR*

Such control is suggested by TOC through DBR: the bottleneck acts as pacemaker (“drum”). For the sake of optimal overall throughput it must always run. In order to prevent it from material starving, a “buffer” is placed right in front of it. The buffer controls material release upstream (“rope”).

2.2 DBR as Reorder Point Problem

Literature discusses DBR related control issues quite extensively (Cox, J. and Schleier, J. 2010; Lödding, 2016) The DBR system can be reduced to a reorder-point system, see Figure 2: the bottleneck consumes at a rate VR, material release is issued when the buffer stock level surpasses the reorder point B, then an amount Q is released. The system must be set up such that the accumulated consumption during lead-time DZ remains manageable. For all practical purposes, $Q \geq B$ must always hold, otherwise the system would constantly be starving.

Dimensioning of such systems, in particular, with respect to B have been analyzed and discussed extensively, for instance with the base stock model (Hopp and

Spearman, 2011). Other aspects as for instance, the proper release point location, or the optimum release amount are discussed in literature too (Lödding, 2016). In particular, this requires decisions with respect to logistical positioning for optimum performance, where "optimum" depends on the target variable selected.

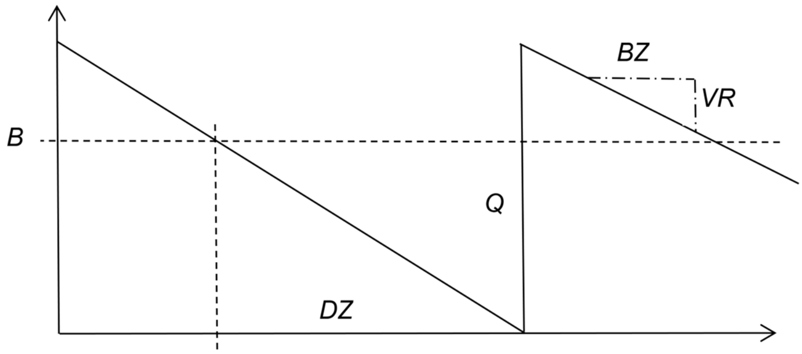

Figure 2: *Reorder Point System*

2.3 Consumption Variation: *VR* or *BZ*

A major difference between the reorder point discussion in literature (Hopp and Spearman, 2011; Chopra and Meindl, 2016) and production reality is that literature suggests use of a consumption rate *VR* at a fixed cycle time *BZ* as described in Figure 2. All practical applications on the shop floor will assume a processing time *BZ* per volume produced (which here is considered one piece at a time only) though. The interpretations for both are quite different:

- In the first case, *DZ* is an amount of intervals, and in each interval, a batch of parts is consumed.
- In the latter case, *DZ* is a frequency, at every interval, one part is consumed.

In static condition, the two can easily be derived from one another as reciprocal values. This proves not to be correct under uncertainty.

2.4 Performance Parameter: FR or CSL

The problem is further complicated by different performance measures: The major output parameters often discussed in the literature are CSL and FR (Chopra and Meindl, 2016). CSL measures the replenishment yield whereas FR measures demand yield. From a practical perspective, the customer related FR is much more important than the supplier related CSL. Four different cases can be distinguished now: Observation of CSL or FR and variation of *VR* or *BZ*.

First, the typical solution by way of the base stock model for *CSL(B)* at variation of *VR* is analyzed: *VR* is defined by a random variable with expected value μ and standard deviation σ for a given distribution *F(x)*. The parameters describe a pattern for volume; every replenishment cycle is divided into *DZ* fixed intervals of such demand pattern, all of which considered independent from each other. The optimum reorder point *B* is then defined as follows:

$$B = F^{-1}\left(CSL, DZ\,\mu, \sqrt{DZ}\,\sigma\right). \quad (1)$$

Second, the analytical relationship *FR(B)* at variation of *VR* is analyzed. Unfortunately, this is more complicated than the above. In fact, a closed analytical solution for the FR under uncertainty is most often limited to special cases like a simple normal distribution of the variation of consumption *VR* (Hopp and Spearman, 2011). Generally, the following approach is being suggested (Hopp and Spearman, 2011; Zipkin, 2000):

$$FR = 1 - \frac{ESC}{B}, \text{ with } ESC = \int_B^\infty (x - B) f(x) dx, \tag{2}$$

utilizing the Expected Shortage Count (*ESC*), based on the density function *f(x)* for *VR*. This can only be resolved for *B* in special cases.

Though Equation (2) cannot always explicitly be solved, the special case solutions that are available do allow for an interesting observation (Hopp and Spearman, 2011):

- CSL is dependent on replenishment time *DZ* but independent from replenishment volume *Q*, for as long as $Q \geq B$ is met.
- FR is dependent on *Q*, but independent from replenishment time *DZ*. Therefore, any parameter study with respect to *FR* will require variation of *Q* additionally.

Third, *CSL(B)* at variation of *BZ* is analyzed: For production, variation in *BZ* is more common than variation in *VR*. In order to derive a proper relationship between *B* and *CSL* at variation of *BZ* as opposed to *VR* above, the statistical experiment must be redefined. Instead of a volume experiment as with the base stock model, now a time experiment must be conducted: CSL is defined by the probability of an uncertain total processing time for a fixed number of parts *B* not to exceed a given boundary time *DZ*:

$$CSL = 1 - F\left(DZ, B\,\mu, \sqrt{B}\,\sigma\right) \tag{3}$$

with μ and σ now defining the stochastic distribution *F* of *BZ* in time. This is unfortunately not quite as elegant if an optimal value of *B* is sought as no explicit solution is easily available. With some numerical effort, it however does the trick.

The fourth and most interesting combination would describe a relationship between the FR and *B* at any given variation in *BZ*. Impact on FR is much more important as a performance criterion than CSL as it reflects customer orientation as opposed to supplier focus. This is however at this time not available and cries for more research.

2.5 Variation in Supply Lead Time *DZ* and Demand Rate *VR*

In reality, both supply lead-time *DZ* and consumption will vary, regardless of whether defined by *VR* or *BZ*. For the remainder of the article, neither the *BZ* case nor the FR analysis can be investigated further as no solution could be offered even for the more simple case of variation in consumption only.

Otherwise literature offers an easy solution for the combined case (Chopra and Meindl, 2016; Hopp and Spearman, 2011): For a consumption process *VR* described by the expected value μ, and standard deviation σ as above, and a supply process *DZ* described by the expected value ϑ, and standard deviation η, the overall system behavior is described by

$$\mu_{DZ+VR} = \mu * \rho, \text{ and } \sigma_{DZ+VR} = \sqrt{\rho\, \sigma^2 + \mu^2\, \eta^2}. \quad (4)$$

Here, it is assumed that all consumption cycles are independent and stationary. Then, means are multiplicative and variations are additive and the above equation contains two types of variation: $\sqrt{\rho\, \sigma^2}$ describes variation within cycles, whereas $\sqrt{\mu^2\, \eta^2}$ describes variation between cycles.

3 Simulation Approach

The simulation platform Plant Simulation (PS) (Bangsow, 2011) provides a very comprehensive system for event based queueing systems of large magnitude. Here, the order of magnitude is very small. Nonetheless, it proves a very powerful system for stochastic analysis, and complexity arises from the scope, not from the scale.

The general setup is chosen such that a source supplies pallets full of parts, which are unloaded and stored in the DBR buffer piece-wise, and processed subsequently. Now, all necessary combinations of fixed or variable supply amounts, fixed or variable processing times and fixed or variable processing volumes can be studied. The DBR buffer is under constant observation, and new material is released as soon as buffer level falls below the reorder point level. Throughout the paper it is assumed that replenishment is an on/off condition that cannot be activated several times at once, it is finished with the last part from the pending replenishment cycle being fed into inventory. The drum is implemented by way of a high resolution time triggered function which controls the inventory level and issues new pallets from the source as needed.

3.1 Variation of Demand Processing Time *BZ*

The most natural approach to modelling a push-oriented supply chain in PS leads to the case of variable processing time *BZ* as represented by Equation (3), the implementation is shown in Figure 3. Left from buffer *B*, the process is lot-based; on the right hand side from *B*, it is executed piece-wise with step size *BZ*.

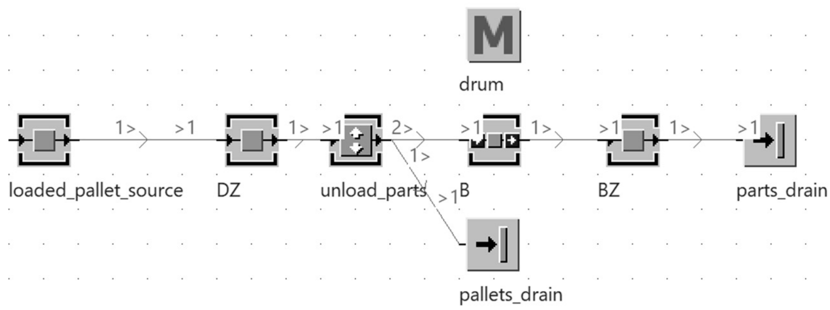

Figure 3: *Implementation with variable processing time BZ*

3.2 Variation of Demand Rate *VR*

In case of a demand rate *VR* variation, the modeling approach is a bit counter-intuitive: The supply side left from *B* is unchanged. The consumption side is very different though. At a fixed time rate, the consumption is batch-oriented with a variable number of parts *VR*. Not always all parts will be available from *B*, then it is a design choice whether or not to work with a backlog list. Such model corresponds with Equations (1) and (2).

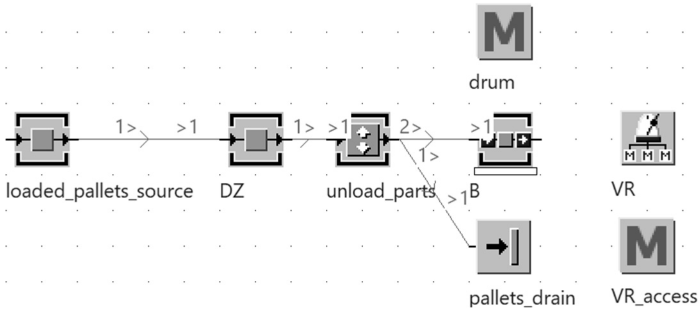

Figure 4: *Implementation with variable consumption rate VR*

3.3 Means to Measuring FR and CSL

In the corresponding PS model, extra provisions must be taken to be able to collect and analyze relevant statistics:

- For every replenishment cycle the simulation must record whether demand was met during this related cycle, thus impacting CSL.
- For any demand requested from *B* it must record if the request could be met sufficiently, thus impacting FR.

Either variable is calculated as the sum of successful attempts over the sum of all attempts.

3.4 Stochastic Variability Modeling

In order to introduce variability into the simulation, a proper distribution form has to be chosen. Logistic problems often show one-dimensional exponential behavior (Hopp and Spearman, 2011), in time. To be able to adjust average and standard deviation separately and still be rather true to reality, the logarithmic normal (lognorm) distribution is employed throughout herein:

$$y = e^{\mu_0 + \sigma_0 x} \text{ with } \mu_0 = \ln \frac{\mu}{\sqrt{\left(\frac{\sigma}{\mu}\right)^2+1}} \text{ and } \sigma_0 = \sqrt{\ln\left[\left(\frac{\sigma}{\mu}\right)^2+1\right]}. \qquad (5)$$

The random variable x is normally distributed with expected value μ_o and standard deviation σ_0. The lognorm distribution is limited to $y > 0$, which is a nice property for simulation. Rather unusual, PS uses a definition of the lognorm with expected value μ and related standard deviation σ as opposed to μ_o and σ_0. These representations can easily be transformed into either space as shown above.

3.5 Parameter Studies

Throughout the analysis, the impact from different reorder levels *B* on CSL and FR is analyzed. For this reason, very many different simulations have to be conducted, sweeping through a whole range of values for *B,* and given the stochastic nature of the model, every simulation must be carried out a few times. For this, PS's ability to perform simulation study experiments has been used extensively for the below parameter studies.

4 Empirical Reorder Point Analysis

Throughout the studies, the reorder point *B* has been analysed in single steps from 1 to 100 pieces. All simulations have been carried out with 50.000 cycles.

4.1 Variation of Demand Processing Time *BZ*

The simulation parameters are chosen as follows: Processing time *BZ* is lognorm distributed with $\mu = \sigma = 1$ cycles per part (cpp), replenishment time is set to *DZ* = 30 cycles, and demand *VR* = 1 piece per cycle (ppc). The results are shown in Figure 5, comparing the simulation results with the analytical solution from Equation (3). The match is not perfect but acceptable given the large number of numerical operations. Thus, the simulation confirms the above model represented by Equation (3).

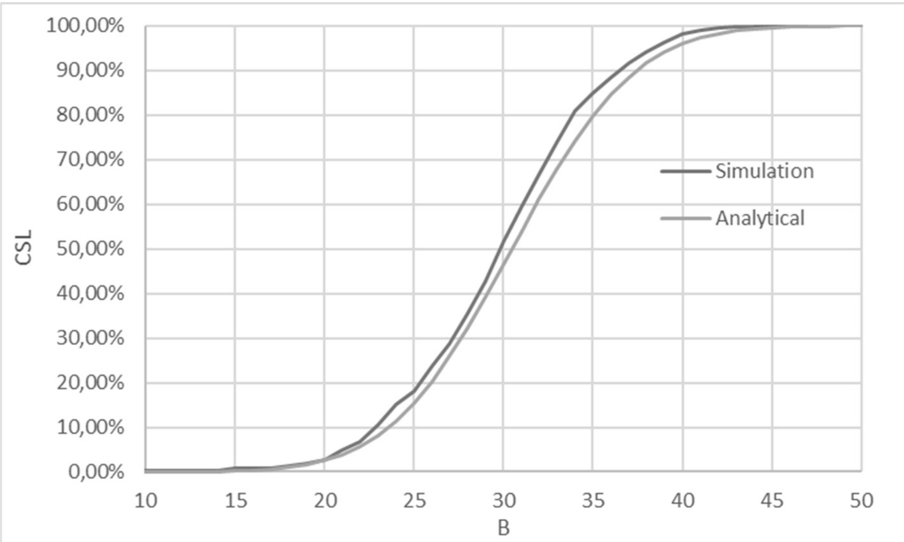

Figure 5: *CSL(B) at Variation of Demand Processing Time BZ*

4.2 Variation of Demand Rate *VR*

First, the impact of *B* on CSL is analyzed, and the following conditions are chosen: *DZ* = 8 cycles, now *VR* is lognorm distributed with $\mu = \sigma =$ 5ppc. The results are shown in Figure 6, comparing simulation results with the results from Equation (1). Here, the difference between simulation and analytical solution is more pronounced. For further validation, an additional analytical solution is included based on the actual random numbers generated throughout the simulation, which turn out to be μ = 5,24ppc and σ = 5,13ppc instead of 5ppc each, which must be attributed to rounding errors or some implementation error within the simulation tool.

Again the results don't show a perfect match but are considered close enough to validate the model from Equation (1), which is not surprising, as it is a welldocumented textbook case anyways. The parameters of the resulting function *CSL(B)* as indicated by Equation (1) reverse-engineered from the simulation results show lognorm behavior with μ = 5,54ppc and σ = 5,32ppc at a RMS = 5 * 10^{-4} over the 100 data points.

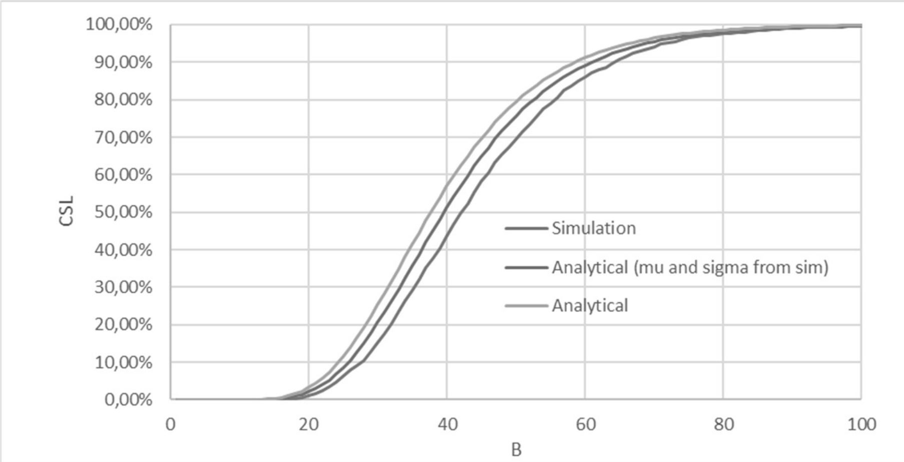

Figure 6: *CSL(B) at Variation of Demand Rate VR*

Next, the more complex *FR(B)* case is investigated for which there was no closed analytical solution of Equation (2). It was resolved numerically using a scientific calculation tool for the given parameters based on a parameter sweep similar to the simulation. The comparison of these semi-analytical results with simulation results is shown in Figure 7.

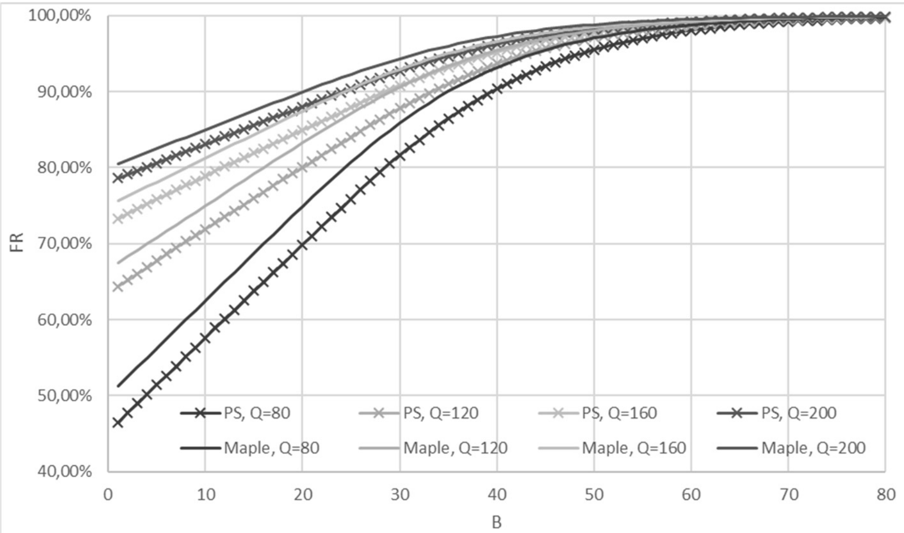

Figure 7: *FR(B) at Variation of Demand Rate VR*

As stated above, now the replenishment volume *Q* must also be taken into account, here it is displayed for a range from 80ppc to 200ppc. Again, the match is not perfect but the trend is confirmed and Equation (2) is validated by the experiment.

4.3 Variation in Supply Lead Time *DZ* and Demand Rate *VZ*

Finally, the case of variation in both replenishment time *DZ* and consumption rate *VR* is analyzed to validate Equation (4). The simulation example from above is being reused: The consumption process *VR* is defined by a lognorm distribution with $\mu = \sigma =$ 5ppc. The supply process is equally described by a lognorm distribution for *DZ* with $\vartheta = \eta =$ 8 cycles. With equation (4) this results in a total system expected value of μ_{ges} = 40ppc and σ_{ges} = 42,43ppc. A comparison of analytical and simulation results is displayed in Figure 8: An approximation of the simulation result *CSL(B)* by a lognorm distribution function results in μ_{ges} = 50,49ppc and σ_{ges} = 61,7ppc at a RMS of $4{,}9 * 10^{-3}$ over the 100 data points. Of course, the results are far from identical, but the trend is confirmed. Further analysis for the source of error revealed that again even the random number generation for the distribution functions wasn't accurate, the rest of the deviation to be attributed to error accumulation. These results do however confirm Equation (4).

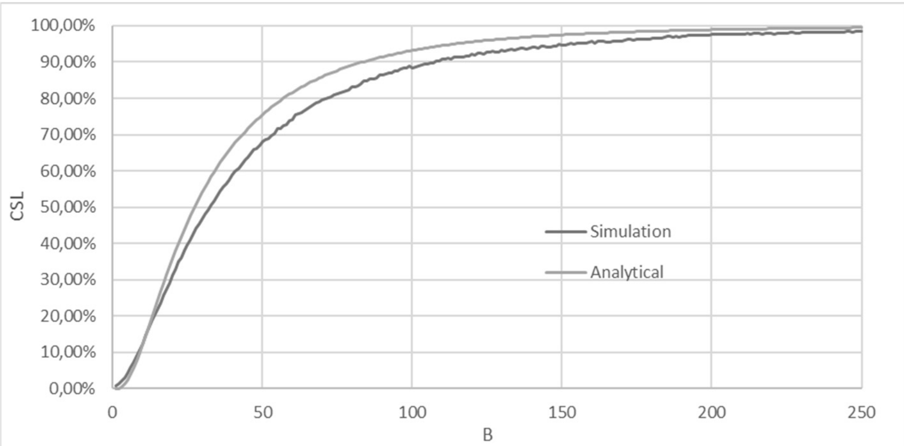

Figure 8: *CSL(B) at Variation of Replenishment time DZ and Demand Rate VR*

All of the corresponding simulation results have shown substantial deviation from the expected. Overall, the trends could be confirmed, but always with certain unease. This is a continuation of observations already made in another study (Rükgauer; Fabeck, 2020). The random number generation of PS is far from precise and it continues to appear that some mysterious rounding error accumulations would always pollute the outcomes.

5 Conclusion

Main focus of this paper was on the relationship between reorder point *B* and the resulting logistical performance parameters when analyzing DBR as reorder point system. Among the many interesting questions arising from such approach, here the relationship between reorder point *B* and the logistical target variables CSL and FR have been investigated analytically and experimentally by way of stochastic simulation.

Literature predominantly focusses on variation in volume *VR* and the related impact on CSL and FR. For the latter, no closed explicit solution is available, but this can be resolved using mathematical solvers Variation in time *BZ* requires more analytical analysis, It is shown that one is not the reciprocal of the other but requires a completely different stochastic experiment setup. At this point, only a solution with respect to CSL is available. In reality, not only consumption will vary, but also supply, in particular, replenishment time *DZ*. Literature offers a nice approach to combine *VR* and *DZ*, for the combination of *BZ* and *DZ*, or even *BZ*, *VR* and *BZ*, so far no solution is available.

Two simulation models have been developed to validate the relationships identified. Simulation of the *VR* case requires an approach that seems almost counter-intuitive. With some numerical error margin, all formulae have been verified using the simulation models.

Based on this work, some solutions are available to the practitioner trying to accomplish desired logistical performance by selecting a proper reorder point *B*. A lot of questions remain unanswered though and call for more research, in particular an extension of the models for the most practically relevant case of *FR(B)* at variation of *BZ*, and the all-encompassing "super-models" describing *CSL(B)* and even more importantly, *FR(B)* at full variation of all of *BZ*, *VR*, and *DZ* at once.

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